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About Verification of Discrete-Continual Finite Element Method of Structural Analysis. Part 2: Three-Dimensional Problems

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Abstract

This paper is devoted to verification of so-called discrete-continual finite element method (DCFEM) of structural analysis. Three-dimensional problems of structural analysis are under consideration. Formulation of the problem of three-dimensional theory of elasticity for structure with piecewise constant physical and geometrical parameters along so-called its basic direction, solutions obtained by DCFEM and finite element method (FEM), their comparison are presented. DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can't be adequately taken into account by the standard FEM.

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Keywords: discrete-continual finite element method; structural analysis; multipoint boundary problem; three-dimensional problems; verification

1. Operational formulation of multipoint boundary problem of three-dimensional structural analysis

Let x_3 be "basic" direction" while physical and geometrical parameters of structure can be changed arbitrarily along x_1 and x_2. Operational formulation of resultant multipoint boundary problem of three-dimensional theory of elasticity at extended domain, embordering considering structure, within DCFEM has the form:

$$\begin{cases} \overline{U}_k' = \widetilde{L}_k \overline{U}_k + \overline{S}_k, \quad x_3 \in (x_{3,k}^b, x_{3,k+1}^b), \quad k = 1, \dots, n_k - 1 \\ \widetilde{B}_k^- \overline{U}_{k-1}(x_{3,k}^b - 0) + \widetilde{B}_k^+ \overline{U}_k(x_{3,k}^b + 0) = \overline{\widetilde{g}}_k^- + \overline{\widetilde{g}}_k^+, \quad k = 2, \dots, n_k - 1 \\ \widetilde{B}_1^+ \overline{U}_1(x_{3,1}^b + 0) + \widetilde{B}_{n_k}^- \overline{U}_{n_k - 1}(x_{3,n_k}^b - 0) = \overline{\widetilde{g}}_1^+ + \overline{\widetilde{g}}_{n_k}^-, \end{cases} \tag{1}$$

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$$\widetilde{L}_k = \begin{bmatrix} 0 & E \\ L_{k,vv}^{-1}(L_{k,uu}+C_k) & L_{k,vv}^{-1}\widetilde{L}_{k,uv} \end{bmatrix}; \quad \overline{S}_k = -\begin{bmatrix} 0 \\ L_{k,vv}^{-1}\overline{\overline{F}}_k \end{bmatrix}; \quad \overline{U}_k = \begin{bmatrix} \overline{u}_k \\ \overline{v}_k \end{bmatrix}; \quad \overline{v}_k = \partial_3 \overline{u}_k; \quad \overline{U}'_k = \partial_3 \overline{U}_k; \tag{2}$$

$$L_{k,vv} = \begin{bmatrix} \overline{\mu}_k & 0 & 0 \\ 0 & \overline{\mu}_k & 0 \\ 0 & 0 & \overline{\lambda}_k + 2\overline{\mu}_k \end{bmatrix}; \quad L_{k,uv} = \begin{bmatrix} 0 & 0 & \partial_1^* \overline{\lambda}_k \\ 0 & 0 & \partial_2^* \overline{\lambda}_k \\ \partial_1^* \overline{\mu}_k & \partial_2^* \overline{\mu}_k & 0 \end{bmatrix} \partial_3; \quad \widetilde{L}_{k,uv} = L_{k,uv} - L_{k,vu}; \quad L_{k,vu} = L_{k,uv}^*; \quad L_{k,vu} = L_{k,uv}^*; \tag{3}$$

$$L_{k,uu} = \sum_{j=1}^{2} \partial_j^* \overline{\mu}_k \partial_j \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \partial_1^* \overline{\mu}_k \partial_1 & \partial_2^* \overline{\mu}_k \partial_1 & 0 \\ \partial_1^* \overline{\mu}_k \partial_2 & \partial_2^* \overline{\mu}_k \partial_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \partial_1^* \overline{\lambda}_k \partial_1 & \partial_1^* \overline{\lambda}_k \partial_2 & 0 \\ \partial_2^* \overline{\lambda}_k \partial_1 & \partial_2^* \overline{\lambda}_k \partial_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad C_k = (\theta_k + \delta_{\Gamma,k}) \begin{bmatrix} c_{k,1} & 0 & 0 \\ 0 & c_{k,2} & 0 \\ 0 & 0 & c_{k,3} \end{bmatrix}; \tag{4}$$

$$\overline{\overline{F}}_k = \theta_k \overline{F}_k + \delta_{\Gamma,k} \overline{f}_k; \quad \theta_k(x) = \begin{cases} 1, & x \in \Omega_k \\ 0, & x \notin \Omega_k; \end{cases} \quad \delta_{\Gamma,k}(x) = \frac{\partial \theta_k}{\partial \overline{n}_k}; \quad \delta'_{\Gamma,k}(x) = \frac{\partial \delta_{\Gamma,k}}{\partial \overline{n}_k}; \tag{5}$$

Ω is the domain, occupied by structure; $x = (x_1, x_2, x_3)$; x_1, x_2, x_3 are coordinates (x_3 corresponds to basic dimension); $x_{3,k}^b$, $k = 1, ..., n_k$ are coordinates of boundary cross-sections of structure (in particular, coordinates of cross-sections with discontinuities of the first kind of physical and geometrical parameters of structure; l_3 is the length of structure along basic dimension, $x_3 \in [0, l_3]$; Ω_k, $k = 1, ..., n_k - 1$ are corresponding fragments of domain Ω with boundaries Γ_k, obtained by separation from domain Ω by cross-sections $x_3 = x_{3,k}^b$ and $x_3 = x_{3,k+1}^b$; ω_k, $k = 1, ..., n_k - 1$ are extended domains, embordering fragments Ω_k, $k = 1, ..., n_k - 1$; $\theta_k = \theta_k(x_1, x_2, x_3)$ is the characteristic function of domain Ω_k; $\delta_{\Gamma,k} = \delta_{\Gamma,k}(x_1, x_2, x_3)$ is the delta-function of border $\Gamma_k = \partial\Omega_k$; $\overline{n}_k = [\, n_{k,1} \;\; n_{k,2} \;\; n_{k,3} \,]^T$ is unit normal vector of domain boundary $\Gamma_k = \partial\Omega_k$; $\overline{u}_k$, $k = 1, ..., n_k - 1$ is the unknown vector of displacements in domain Ω_k; $\widetilde{B}_k^-, \widetilde{B}_k^+$, $k = 2, ..., n_k - 1$, $\widetilde{B}_1^+$, $\widetilde{B}_{n_k}^-$ are matrices (operators) of boundary conditions of the sixth order (x_3-independent); $\overline{\overline{g}}_k^-, \overline{\overline{g}}_k^+$, $k = 2, ..., n_k - 1$, $\overline{\overline{g}}_1^+$, $\overline{\overline{g}}_{n_k}^-$ are right-side vectors of boundary conditions of the sixth order (x_3-independent); $\overline{\overline{F}}_k$ is the right-side vector in domain Ω_k; $\overline{F}_k$ is the vector of body forces in domain Ω_k; $\overline{f}_k$ is the boundary traction vector in domain Ω_k; $\overline{\lambda}_k$, $\overline{\mu}_k$ are Lame coefficients of material in domain Ω_k; C_k is the matrix of elastic parameters of the supports (if any); $c_{k,i}$ is the coefficient of resistance in the direction of the axis Ox_i; $\partial_k = \partial / \partial x_k$, $\partial_k^* = -\partial / \partial x_k$, $k = 1, 2$; $\overline{v}_k = \partial_3 \overline{u}_k = \overline{u}'_k$; $\overline{v}'_k = \partial_3 \overline{v}_k$.

2. Discrete-continual formulation of multipoint boundary problem of three-dimensional structural analysis

DCFEM presupposes finite element approximation of extended domain along directions of structure perpendicular to the basic direction, while along basic direction problem remain continual (thus extended domain is divided into discrete-continual finite elements). Resulting multipoint boundary problem for the first-order system of ordinary differential equations with piecewise-constant coefficients within DCFEM [1, 2] has the form:

$$\begin{cases} \overline{y}_k^{(1)} - A_k \overline{y}_k = \overline{f}_k, \quad x_3 \in (x_{3,k}^b, x_{3,k+1}^b), \quad k = 1, 2, ..., n_k - 1 \\ B_k^- \overline{y}_k(x_{3,k}^b - 0) + B_k^+ \overline{y}_k(x_{3,k}^b + 0) = \overline{g}_k^- + \overline{g}_k^+, \quad k = 2, ..., n_k - 1 \\ B_1^+ \, \overline{y}_k(x_{3,1}^b + 0) + B_{n_k}^- \, \overline{y}_k(x_{3,n_k}^b - 0) = \overline{g}_1^+ + \overline{g}_{n_k}^-, \end{cases} \tag{6}$$

where A_k, $k = 1, 2, ..., n_k - 1$ are matrices of constant coefficients of order $n = 6N_1N_2$ (discrete analogs of operators $\widetilde{L}_k$, $k = 1, 2, ..., n_k - 1$); $\overline{f}_k$, $k = 1, 2, ..., n_k - 1$ are vectors of size $n = 6N_1N_2$ (discrete analogs of vector functions $\overline{S}_k$, $k = 1, 2, ..., n_k - 1$); $N_1 - 1$ is the number of elements along x_1; $N_2 - 1$ is the number of elements along x_2.

$$\bar{y}_k = \bar{y}_k(x_3) = [\,\bar{u}_k^T(x_3) \quad \bar{v}_k^T(x_3)\,]^T ; \tag{7}$$

$$\bar{u}_k = \bar{u}_k(x_3) = [\,(\bar{u}_n^{(k,1,1)})^T \quad (\bar{u}_n^{(k,2,1)})^T \quad \ldots \quad (\bar{u}_n^{(k,N_1,1)})^T \quad \ldots \\ \ldots \quad (\bar{u}_n^{(k,1,2)})^T \quad (\bar{u}_n^{(k,2,2)})^T \quad \ldots \quad (\bar{u}_n^{(k,N_1,2)})^T \quad \ldots \quad (\bar{u}_n^{(k,1,N_1)})^T \quad (\bar{u}_n^{(k,2,N_1)})^T \quad \ldots \quad (\bar{u}_n^{(k,N_1,N_2)})^T\,]^T ; \tag{8}$$

$$\bar{v}_k = \bar{v}_k(x_3) = [\,(\bar{v}_n^{(k,1,1)})^T \quad (\bar{v}_n^{(k,2,1)})^T \quad \ldots \quad (\bar{v}_n^{(k,N_1,1)})^T \quad \ldots \\ \ldots \quad (\bar{v}_n^{(k,1,2)})^T \quad (\bar{v}_n^{(k,2,2)})^T \quad \ldots \quad (\bar{v}_n^{(k,N_1,2)})^T \quad \ldots \quad (\bar{v}_n^{(k,1,N_1)})^T \quad (\bar{v}_n^{(k,2,N_1)})^T \quad \ldots \quad (\bar{v}_n^{(k,N_1,N_2)})^T\,]^T ; \tag{9}$$

$$\bar{u}_n^{(k,p,q)} = \bar{u}_n^{(k,p,q)}(x_3) = [\,u_1^{(k,p,q)} \quad u_2^{(k,p,q)} \quad u_3^{(k,p,q)}\,]^T, \quad p = 1,2,\ldots,N_1, \quad q = 1,2,\ldots,N_2 ; \tag{10}$$

$$\bar{v}_n^{(k,p,q)} = \bar{v}_n^{(k,p,q)}(x_3) = [\,v_1^{(k,p,q)} \quad v_2^{(k,p,q)} \quad v_3^{(k,p,q)}\,]^T, \quad p = 1,2,\ldots,N_1, \quad q = 1,2,\ldots,N_2 ; \tag{11}$$

$u_i^{(k,p,q)} = u_i^{(k,p,q)}(x_3),\ \ p = 1,2,\ldots,N_1,\ \ q = 1,2,\ldots,N_2,\ \ k = 1,2,\ldots,n_k - 1$ are functions, which define component of displacement u_i in the node with coordinate (x_1^p, x_2^q, x_3) in the interval $x_3 \in (x_{3,k}^b, x_{3,k+1}^b)$.

Solution of problem (6) is accentuated by numerous factors. They include boundary effects (stiff systems) and considerable number of differential equations (several thousands). Moreover, matrices of coefficients of a system normally have eigenvalues of opposite signs and corresponding Jordan matrices are not diagonal. Special method of solution of multipoint boundary problems for systems of ordinary differential equations with piecewise constant coefficients in structural analysis has been developed [1]. Not only does it overcome all difficulties mentioned above but its major peculiarities also include universality, computer-oriented algorithm, computational stability, optimal conditionality of resultant systems and partial Jordan decomposition of matrix of coefficient, eliminating necessity of calculation of root (principal) vectors [1].

3. Numerical Sample

DCFEM, considering in the distinctive paper, has been realized in software DCFEM3Dpc. Programming environment is Microsoft Visual Studio 2012 Professional and Intel Parallel Studio XE 2013 (Intel Visual Fortran Composer XE 2013). Test, model and practically important problems of structural analysis have been solved with the use of DCFEM3Dpc.

Let's consider three-dimensional structure with rectangular cross sections fixed on two sides ($x_2 = 0$ and $x_2 = L$ with zero displacements $u_1 = u_2 = u_3 = 0$). Length of structure (L) is equal to 600 cm (Figure 1). Height of structure (h_2) is equal to 50 cm, width (h_1) is equal to 50 cm. Additional geometrical parameters (Fig. 1): $a_1 = 150$ cm; $a_2 = 450$ cm. Elastic modules of material for the first ($x_2 \in (0, L/2)$) and the second ($x_2 \in (L/2, L)$) parts of structure (E_1, E_2) are equal to 3000 kN/cm^2 and to 3500 kN/cm^2 respectively. Poisson's ratios of material for the first and the second parts of structure (ν_1, ν_2) are equal to 0.16 and to 0.14 respectively. Structure is loaded by concentrated forces $P_1 = 100$ kN and $P_2 = 100$ kN.

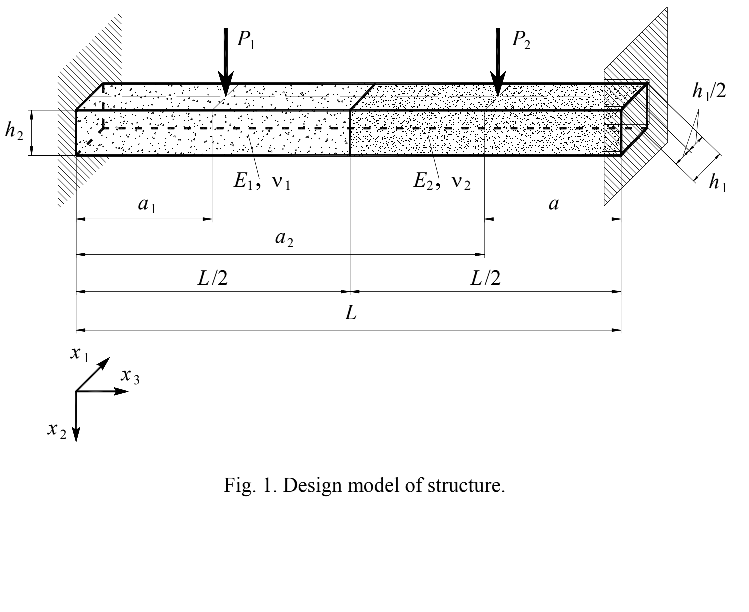


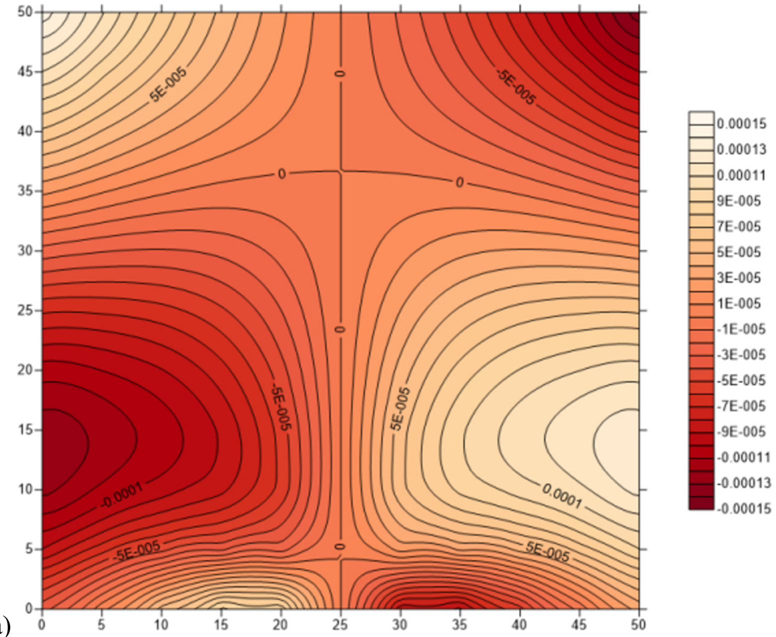
Fig. 1. Design model of structure.

ANSYS Mechanical (ANSYS 15.0) simulation software has been used for solution of problem in terms of FEM.

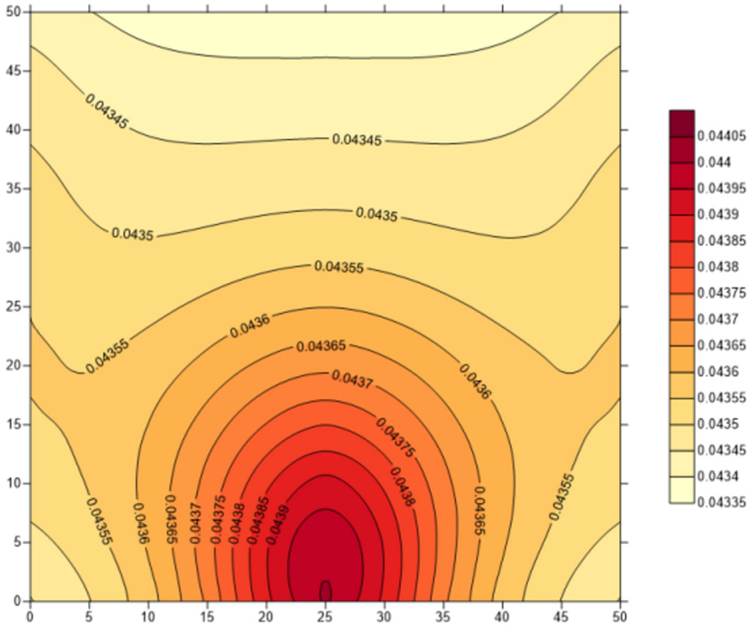
Cartesian coordinate system (x, y, z) has been used. We have $x = x_3$, $y = -x_2$, $z = -x_1$.

Uniform square mesh 60x10x10 (in ANSYS 15.0) have been constructed from SOLID185 finite element [3].

DCFEM3Dpc simulation software has been used for solution of problem in terms of DCFEM. Uniform approximating mesh along x_1 and x_2 includes 10x10 discrete-continual finite elements. Distributions of displacements u_1, u_2, u_3 and stress σ_{11} at cross section $x_3 = 140$ sm are presented at Fig. 2. Comparison of stresses and displacements, obtained by ANSYS Mechanical and DCFEM3Dpc at several cross-sections of deep beam are presented at Fig. 3. Thus, we can conclude that the results of analysis obtained by the ANSYS Mechanical (ANSYS 15.0) and DCFEM3Dpc simulation software generally agree well with each other.



a)



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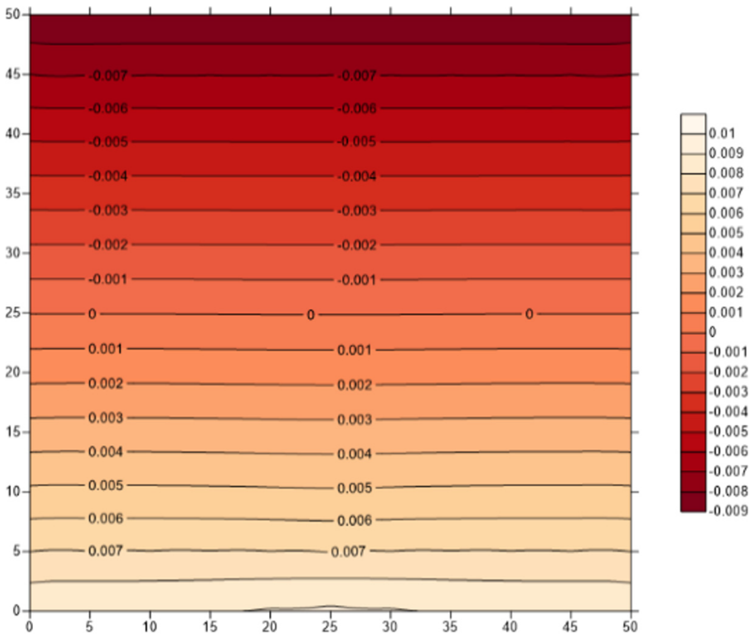

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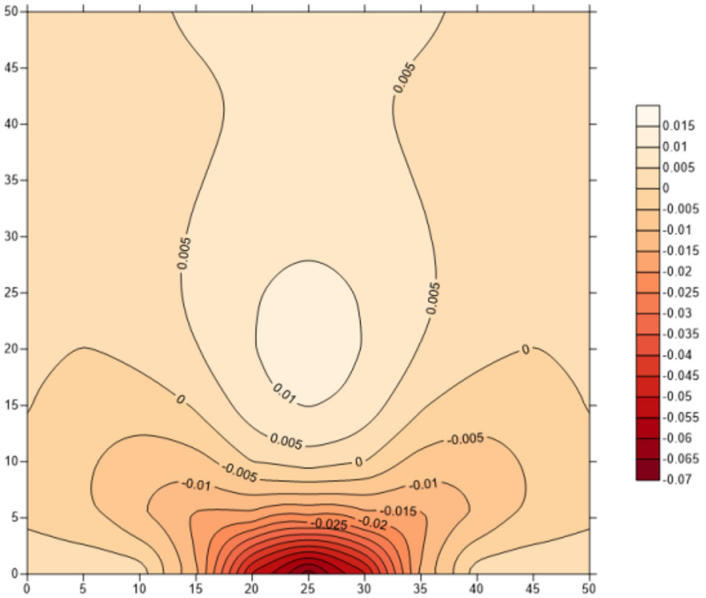

d)

Fig. 2. DCFEM distribution of parameters at cross section $x_3 = 140$ sm: (a) displacements u_1 (cm); displacements u_2 (cm); (c) displacements u_3 (cm); (d) stresses $\sigma_{1,1}$ (kN/cm^2).

It was confirmed that DCFEM is more effective in the most critical, vital, potentially dangerous areas of structure in terms of fracture (areas of the so-called edge effects), where some components of solution are rapidly changing functions and their rate of change in many cases can't be adequately taken into account by the standard finite element method [3, 4].

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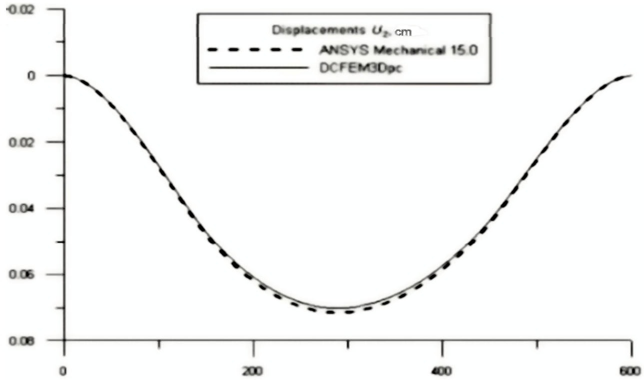

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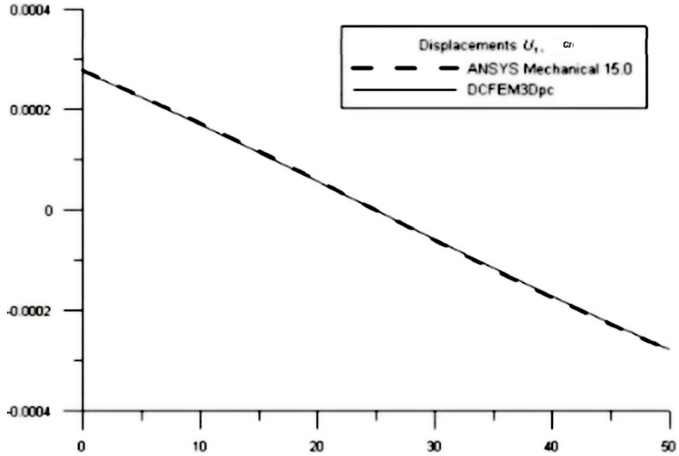

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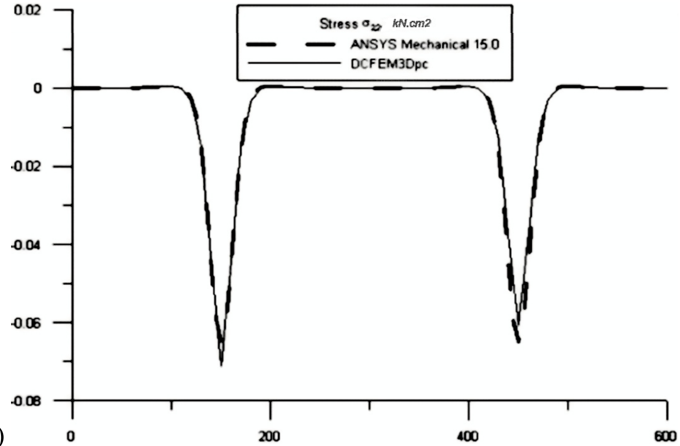

c)

Fig. 3. Comparison of results, obtained by ANSYS and DCFEM3Dpc: (a) distribution of displacements u_2 along x_3 ($x_1 = x_2 = 0$), cm; (b) distribution of displacements u_1 along x_2 ($x_1 = 10$ cm, $x_3 = 30$ cm), (c) distribution of stresses $\sigma_{2,2}$ along x_3 ($x_1 = 25$ cm, $x_2 = 25$ cm).

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